Global existence and finite time blow-up of solutions of a Gierer-Meinhardt system *

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Abstract

We are concerned with the Gierer-Meinhardt system with zero Neumann boundary condition:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u - a_1 u + \frac{u^p}{v^q} + \delta_1(x), & x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v - a_2 v + \frac{v^r}{u^s} + \delta_2(x), & x \in \Omega, \ t > 0, \\
u(x, 0) &= u_0(x), & v(x, 0) = v_0(x), & x \in \Omega,
\end{align*}
\]

where \(p > 1, s > -1, q, r, d_1, d_2, a_1, a_2\) are positive constants, \(\delta_1, \delta_2, u_0, v_0\) are nonnegative smooth functions, \(\Omega \subset \mathbb{R}^d \ (d \geq 1)\) is a bounded smooth.

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domain. We obtain new sufficient conditions for global existence and finite time blow-up of solutions, especially in the critical exponent cases: \(p - 1 = r\) and \(qr = (p - 1)(s + 1)\).

**Keywords:** Gierer-Meinhardt system; Solution; Global existence; Finite time blow-up.

1 Introduction

In this paper, of our concern is the following general Gierer-Meinhardt system

\[
\begin{aligned}
    u_t &= d_1 \Delta u - a_1 u + \frac{u^p}{v^q} + \delta_1(x), \quad x \in \Omega, \ t > 0, \\
    v_t &= d_2 \Delta v - a_2 v + \frac{u^r}{v^s} + \delta_2(x), \quad x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]

(1.1)

where \(p > 1, s > -1, q, r, d_1, d_2, a_1, a_2\) are positive constants, \(\delta_1, \delta_2, u_0, v_0\) are nonnegative smooth functions, \(\Omega \subset \mathbb{R}^d\) is a bounded domain with smooth boundary \(\partial \Omega\) and the space dimension \(d \geq 1\), and \(\nu\) is the outward norm vector.

Following the idea of *diffusion-driven instability* proposed by Turing [41], Gierer and Meinhardt [12] in 1972 introduced the reaction-diffusion system (1.1) to model the pattern formation of spatial tissue structures of hydra in morphogenesis, a biological phenomenon which was discovered by Trembley in 1744 [42]. It is noted that in the original Gierer-Meinhardt model, \(\delta_1\) is a nonnegative constant and \(\delta_2 \equiv 0\); the general Gierer-Meinhardt model (1.1) was proposed in [14].

The Gierer-Meinhardt system (1.1) is one of the most famous models in biological pattern formation and belongs to the activator-inhibitor type. In the past few decades, the Gierer-Meinhardt system (1.1) has received extensive attention in research. The existence and uniqueness of a local solution to (1.1) is a folklore fact of standard parabolic theory; see, for example, [27], for details. Throughout the paper, a solution of (1.1) always means a classical nonnegative one. From the pure mathematical point of view, a fundamental question is the global existence of solution to (1.1). By a global solution of (1.1) we mean that its maximum existence time \(T_{max} = \infty\), and by a blow-up solution \((u, v)\) we mean that its maximum existence time \(T_{max} < \infty\) and \(\lim_{t \to T_{max}} \sup_{x \in \Omega} (u(x, t) + v(x, t)) = \infty\). In the paper, unless otherwise stated, we assume that the initial data \((u_0, v_0)\) satisfy

\[
u_0(x) \geq 0, \quad v_0(x) > 0, \quad \forall x \in \overline{\Omega}.
\]

According to the strong maximum principle for parabolic equations, the solution