USING EFFECTIVE BOUNDARY CONDITIONS TO MODEL FAST DIFFUSION ON A ROAD IN A LARGE FIELD

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Abstract. We consider a logistic diffusion equation on the plane consisting of two components, a “road” that is parallel to the $x$-axis, and a “field”, in each of which the diffusion rate differs significantly. Compared to the size of the field, the width $\delta$ of the road is assumed to be small. Thus in this diffusion equation multiple scales appear in two places: the spatial variable and the diffusion parameter. Such an equation is not easy to solve numerically, and it is not easy to see the effects of the road. Recently, Berestycki, Roquejoffre and Rossi provide a model which is meant to resolve these issues. In this paper we first use the idea of effective boundary conditions (EBCs) to propose, rigorously, a different model: we study the limit of the solution of the original logistic equation as $\delta \to 0$, obtaining a limiting model, in which the road now is the $x$-axis with EBCs imposed on it. This effective problem has no multiple scales and hence should be easier to solve numerically. Moreover, to see the effects of the road, we further investigate the asymptotic propagation speed of the effective model, showing that the road indeed enhances the spreading speed along its direction, provided that the diffusion rate on the road is of order $O(\delta^{-1})$.

Key words. Logistic diffusion equation; Asymptotic behavior; Effective boundary condition; Propagation speed; Comparison principle; Regularity.

AMS subject classifications. 35K57, 35B40, 35Q92.

1. Introduction. To understand the motivation of this paper, the reader should have the mental picture of the following scenario (among many others): imagine that a species of animals diffuses in a large nature reserve with a road running through (see Figure 1). The road is very narrow, compared to the big size of the reserve; the diffusion rate of the animals is larger in the road than in the field. Thus multiple scales are present not only in the spatial variable, but also in the physical parameter. A full model for the dynamics of the animal population would be a diffusion equation that includes all these scales, and therefore is cumbersome and difficult to solve numerically (very fine meshes are needed in the road). A natural way to simplify the full model is to think of the road as being widthless, and then impose certain conditions on it.

In [5], Berestycki, Roquejoffre and Rossi provide such a model: the road in their model is the widthless $x$-axis, off which is the field; the field is so large that it is assumed to be the entire plane; in the field the population obeys Fisher-KPP equation, on the road the population satisfies a diffusion equation with the source term coming from the exchange of populations between the road and the field; moreover, a Robin boundary condition is also imposed on the road, reflecting the aforementioned exchange of the populations. The most important phenomenon exhibited by their model is that large diffusion rate on the road enhances the asymptotic propagation speed in the field, at least along the direction of the road.

In this paper, we use the idea of effective boundary conditions to propose a

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